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Note

Non-repetitive colorings of infinite sets

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Abstract

In this paper we investigate colorings of sets avoiding similarly colored subsets. If S is an arbitrary colored set and $\mathscr{T}$ is a fixed family of bijections of S to itself, then two subsets $A, B \subseteq S$ are said to be colored *similarly* with respect to $\mathscr{T}$, if there exists a transformation $t \in \mathscr{T}$ mapping A onto B, and preserving a coloring of A. This general setting covers some well-known topics such as *non-repetitive* sequences of Thue or the famous Hadwiger–Nelson problem on unit distances in Euclidean spaces. Our main theorem of this paper concerns arbitrary infinite sets, however, the most striking consequences are obtained for the case of Euclidean spaces. For instance, there exist 2-colorings of $\mathbb{R}^n$ with no two different line segments colored similarly, with respect to translations. The method is based on the principle of induction, hence it is not constructive in general, and the problem of explicit constructions arises naturally. We give two such examples of non-repetitive colorings of the sets $\mathbb{R}$ and $\mathbb{Q}$, with respect to translations. In conclusion of the paper we discuss possible generalizations and pose two open problems.

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1. Introduction

In this paper we study repetitions of patterns in 2-colorings of infinite sets. Suppose, for example, that we want to paint the line so as to avoid identically colored segments. Two given segments A and B are colored *identically*, if there exists a *translation* t, transforming A onto B, and preserving colors of all points of A. More precisely, if f denotes the coloring of the line, then $f(a) = f(t(a))$, for all $a \in A$. As a rather special

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case of our main theorem of this paper we obtain that *there are* 2*-colorings of the line in which no two different segments are colored identically*.

There are many problems of a similar kind, especially in Euclidean Ramsey Theory, some of which are quite famous. Perhaps the most intriguing one is the Hadwiger–Nelson problem on unit distances in Euclidean spaces. A k-coloring $f:\mathbb{R}^n \to \{0,1,\ldots,k-1\}$ *avoids* unit distances if $f(x) \neq f(y)$ for any two points x, y distance one apart. What is the minimal number of colors needed for such a coloring? Sometimes it is called *the chromatic number of* $\mathbb{R}^n$ and is denoted by $\chi(\mathbb{R}^n)$. It is easy to see that $\chi(\mathbb{R}) = 2$; simply, divide the line into half-open intervals of the form $[n, n+1)$, $n \in \mathbb{Z}$, and then paint them alternatively red and blue. Surprisingly, for higher dimensions the problem is much harder and is still open, even for the plane. By simple constructions one can get quickly the inequalities $4 \leqslant \chi(\mathbb{R}^2) \leqslant 7$, but finding the exact value is certainly one of the most challenging problems of discrete geometry (see [4]).

Many variations on the distance avoiding theme are possible. For instance, suppose that we want to avoid more different distances with as few colors as possible. As the theorem of Erdős et al. [10] asserts, there exists a set S of cardinality 2^{ω} and a 2-coloring of $\mathbb{R}$ such that $f(x) \neq f(y)$, if the distance between x and y belongs to S. But, on the other hand, such an S cannot have positive measure.

Much stronger avoidability can be achieved having more colors at a disposal. In 1943, Erdős and Kakutani [11] proved, that there exists an ω-coloring of $\mathbb{R}$ in which *any* given distance appears *at most once*, that is, if $f(x) = f(y)$, $x \neq y$, then $f(x+t) \neq f(y+t)$, for every $t \neq 0$. Unfortunately, this incredible property cannot be established without assuming the truth of the Continuum Hypothesis. Their result was then generalized for the plane by Davies [7], and finally, for all finite dimensional spaces, by Kunen [14].

Another fascinating discovery which was an inspiration for our studies goes back to Thue, and this time concerns discrete sets. In 1906 Thue proved that there exist infinite sequences over *three* symbols in which no *two* adjacent blocks are exactly the same (see [2,16,19]). It is seen easily that two symbols do not suffice for such a peculiar property, but, on the other hand, it is possible to build an infinite *binary* sequence without *three* consecutive identical blocks. Such *non-repetitive* sequences have found a dozen of applications in such diverse areas as group theory, ergodic theory, number theory or formal language theory (see [2,3,5,6,16]). For instance, they play a role of fundamental importance in the solution of the famous Burnside conjecture for groups (see [15]). The method invented by Thue is constructive and reduces to finding an appropriate *substitution* over a given alphabet of symbols. For example, the substitution

$$a \to abcab$$

$$b \to acabcb$$

$$c \to acbcacb$$

preserves the property of non-repetitiveness on the set of finite strings over an alphabet $\{a,b,c\}$. In consequence, there are arbitrarily long finite non-repetitive strings, and the existence of an infinite sequence without repetitions follows, via the König's Infinity Lemma (see [2]).

In the course of generalizing the theorems of Thue many beautiful results were found. In [2] Bean et al. introduced *square-free* colorings of general linear orders and established the existence of such 2-colorings of $\mathbb{R}$ and $\mathbb{Q}$. In case of $\mathbb{R}$ this means that for any two points $x < y$ there exists $t \in [x, y)$, such that $f(x+t) \neq f(y+t)$. Larson et al. [15] proved soon that three colors are always sufficient for a square-free coloring of an arbitrary ordinal α.

Our main purpose in this paper is to prove a general existence theorem (Theorem 2) on colorings of infinite sets, allowing to deduce some, rather unexpected, results of similar type. We conclude this introduction with one such, particularly striking, example, extending the theorem of Bean et al. on square-free colorings of the reals.

Theorem 1. *There exists a* 2*-coloring f of the real line*, *such that given any countable configuration* $A \subset \mathbb{R}$, *any function* $p : A \to \{0, 1\}$, (*a color pattern*), *and any* $\varepsilon > 0$, *there exists* $0 \leqslant t < \varepsilon$, *such that* $f(a+t) = p(a)$, *for all* $a \in A$.

In fact, for A running through two-element subsets of the line, we obtain the result mentioned at the beginning, which asserts stronger property than the square-freeness of f.

Our method is based mainly on transfinite induction, thus most results are non-constructive. So, the question of explicit colorings arise naturally and we give two such constructions in Section 3, for the sets of real and rational numbers. The first one, a genially simple function based on natural logarithm, has been found by Rote [18]. The proof that it does the job is easy, however, it relies on the Lindemann–Weierstrass theorem, a deep result of algebraic number theory. The second one, concerning the rational case, uses only elementary arithmetic, but this time the details are rather complicated.

In the last section of the paper we propose some open problems respecting continuous as well as discrete aspects of considered topic.

2. Coloring infinite sets

In this section we prove the main theorem, mentioned in the Introduction, and illustrate it with some simple applications.

Theorem 2. *Let S be an arbitrary infinite set and let* $T : S \times S \to S$ *be a left and right injective mapping. For a fixed* $t \in S$ *denote by* T_t *a mapping of S to itself, defined by* $T_t(x) = T(t, x)$, $x \in S$. *Further, consider an arbitrary family* $\mathscr{F}$ *of subsets of S, such that* $|\mathscr{F}| \leqslant |S|$ *and* $|E| = |S|$, *for any E from* $\mathscr{F}$. *Then there exists a coloring function* $f : S \to \{0, 1\}$ *such that for every subset* $X \subseteq S$, *satisfying* $|S^X| = |S|$, *for every function* $p : X \to \{0, 1\}$, *and for every set* $E \in \mathscr{F}$, *there exists* $t \in E$ *such that* $f(T_t(x)) = p(x)$, *for all* $x \in X$.

Proof. Let $\sigma = |S|$ and suppose that β is the least cardinal satisfying $\sigma^\beta > \sigma$. Clearly, $\beta \leqslant \sigma$. Let $\mathscr{I} = \{X \subseteq S; |X| < \beta\}$ and $\mathscr{R} = \{(X, Y, E) \in \mathscr{I} \times \mathscr{I} \times \mathscr{F}; Y \subseteq X\}$. Note that

$|\mathscr{I}| = |\mathscr{R}| = \sigma$. In fact, if we denote $\mathscr{I}_\alpha = \{X \subseteq S; |X| = \alpha\}$, then $|\mathscr{I}_\alpha| = \sigma^\alpha = \sigma$, for $0 < \alpha < \beta$, and $\bigcup_{\alpha<\beta} \mathscr{I}_\alpha = \mathscr{I}$.

Let ψ be the least ordinal of cardinality σ. We will construct, by the use of induction (transfinite, in case $\sigma > \aleph_0$), a sequence $\{(A_\varphi, B_\varphi)\}_{\varphi<\psi}$ of pairs of subsets of S, satisfying the following conditions.

(1_φ) $A_\varphi \cap B_\varphi = \emptyset$;
(2_φ) $A_{\varphi_1} \subset A_\varphi, B_{\varphi_1} \subset B_\varphi$, for $\varphi_1 < \varphi$;
(3_φ) $|A_\varphi \cup B_\varphi| \leqslant \max\{\beta, |\varphi|\}$ if $\beta < \sigma$, and $|A_\varphi \cup B_\varphi| < \sigma$ if $\beta = \sigma$.

Let $g : [0,\psi) \to \mathscr{R}$ be a bijection well-ordering the set $\mathscr{R}$. Denoting $g(0) = (X_0, Y_0, E_0)$, we can take $A_0 = T_{t_0}(Y_0)$ and $B_0 = T_{t_0}(X_0 \setminus Y_0)$, with some $t_0 \in E_0$, as an initial step. Next, suppose that $0 < \xi < \psi$ and that we have already constructed the sets A_φ, B_φ, $\varphi < \xi$, satisfying conditions (1_φ), (2_φ) and (3_φ). We are going to construct A_ξ and B_ξ. Denote $w_\varphi = |A_\varphi \cup B_\varphi|$ and $\omega(\xi) = \sum_{\varphi<\xi} w_\varphi$. If $\beta < \sigma$ then we have

$$\omega(\xi) \leqslant |\xi| \max\{\beta, |\xi|\} = \max\{\beta, |\xi|\}.$$

If $\beta = \sigma$ then we obtain

$$\sigma^{\omega(\xi)} = \prod_{\varphi<\xi} \sigma^{w_\varphi} = \prod_{\varphi<\xi} \sigma = \sigma^{|\xi|} = \sigma.$$

Hence, $\omega(\xi) < \beta$.

Now, denoting $g(\xi) = (X, Y, E)$, it is seen easily that the set $W = \{t \in E; T_t(X) \cap \bigcup_{\varphi<\xi} (A_\varphi \cup B_\varphi) = \emptyset\}$ is non-empty. In fact, we have $E \setminus W = \bigcup_{x\in X} \{t \in E; T_t(x) \in \bigcup_{\varphi<\xi} (A_\varphi \cup B_\varphi)\}$, so, $|E \setminus W| \leqslant |X| \sum_{\varphi<\xi} w_\varphi < \sigma$. Finally, pick $t \in W$ and set

$$A_\xi = \bigcup_{\varphi<\xi} A_\varphi \cup T_t(Y) \quad \text{and} \quad B_\xi = \bigcup_{\varphi<\xi} B_\varphi \cup T_t(X \setminus Y).$$

Clearly, $A_\xi, B_\xi \subseteq S$, $A_\xi \cap B_\xi = \emptyset$ and $A_\varphi \subset A_\xi, B_\varphi \subset B_\xi$, for all $\varphi < \xi$. Moreover,

$$|A_\xi \cup B_\xi| \leqslant \sum_{\varphi<\xi} (w_\varphi + \beta) \leqslant \max\{\beta, |\xi|\},$$

if $\beta < \sigma$, and

$$|A_\xi \cup B_\xi| \leqslant \sum_{\varphi<\xi} (w_\varphi + |X|) < \sigma,$$

if $\beta = \sigma$. Thus, (A_ξ, B_ξ) satisfies conditions (1_ξ)–(3_ξ) and the inductive step is complete.

Consider now the sets $A = \bigcup_{\varphi<\psi} A_\varphi$ and $B = \bigcup_{\varphi<\psi} B_\varphi$. The definition of a coloring function f with the desired properties is based on the set A. Put $f(x) = 1$ if $x \in A$, and $f(x) = 0$, otherwise. Then, given a set X, a function $p : X \to \{0, 1\}$ and some $E \in \mathscr{F}$, there is a corresponding triple $(X, Y, E) = g(\xi)$ in $\mathscr{R}$, with $\xi < \psi$, where $Y = p^{-1}(1)$.

It follows from the construction of the sets A_ξ and B_ξ that there exists $t \in E$ such that $T_t(Y) \subset A_\xi \subset A$ and $T_t(X \setminus Y) \subset B_\xi \subset B$. But the sets A and B are disjoint, so $A \cap T_t(X) = T_t(Y)$, which proves the assertion of Theorem 2. $\square$

To obtain Theorem 1, formulated in the Introduction, put $S = \mathbb{R}$, $T_t(x) = t + x$ and $\mathscr{F} = \{[0, \varepsilon); \varepsilon > 0\}$. Actually, a similar result can be deduced from Theorem 2 for any Euclidean space, by considering a family of balls of positive radius centered at the origin. In particular, there exists a 2-coloring of the plane such that no two different translated copies of the same square are colored similarly.

3. Explicit constructions

The first example we present here has been discovered by Günter Rote [18] and concerns a non-repetitive coloring of the set $\mathbb{R}$ of real numbers. To prove that it satisfies the desired property we will need the famous result from algebraic number theory, known as the Lindemann–Weierstrass theorem (see [1]). It asserts the linear independence of algebraic powers of e over the field of algebraic numbers. More precisely, the equation $a_1 e^{b_1} + \cdots + a_n e^{b_n} = 0$ cannot hold, if $a_1, \ldots, a_n$ are non-zero algebraic numbers and $b_1, \ldots, b_n$ are pairwise distinct algebraic numbers.

Theorem 3 (Rote [18]). *Define a coloring function* $f : \mathbb{R} \to \{0, 1\}$ *by* $f(x) = 0$ *if* $\ln|x| \in \mathbb{Q}$, *and* $f(x) = 1$ *in all other cases. Then, for any given* $\varepsilon > 0$ *and any real numbers* $x < y$ *there exists* $0 \leqslant t < \varepsilon$, *such that* $f(x + t) \neq f(y + t)$. *In other words, no two different line segments are colored similarly with respect to translations.*

Proof. The signs of considered numbers do not matter, so, assume, only for convenience, that $0 < x < y$. If colors of x and y agree, shift them slightly to get $x + t_1 = e^{q_1}$, where $0 \leqslant t_1 < \varepsilon$ and $q_1 \in \mathbb{Q}$. Then $f(x + t_1) = 0$, by the definition of f. If at the same time $f(y + t_1) = 0$, then there must exist a rational number $q_2 \neq q_1$, such that $y + t_1 = e^{q_2}$. In such a bad situation we have to make another shift by some $t_1 < t_2 < \varepsilon$ to get $x + t_2 = e^{q_3}$, for some $q_3 \in \mathbb{Q}$, different from q_1 and q_2. If it would again happened that $y + t_2 = e^{q_4}$ is a rational power of e, then we will get the equality $e^{q_1} - e^{q_2} = e^{q_3} - e^{q_4}$, with all q_i *different* rational numbers. However, this contradicts the Lindemann–Weierstrass theorem, hence the proof is complete. $\square$

In the second example we present a function with an analogous property, coloring the set of rationals $\mathbb{Q}$. It uses only simple properties of prime numbers. However, as it is often the case, an elementary situation causes more troubles. Note that the function from previous theorem is constant on rationals, hence cannot be applied here.

To formulate the result we have to adopt some notation. As usually, by $\mathbb{N}$ and $\mathbb{Z}$ we mean the set of positive integers and the set of all integers, respectively. The symbol $\mathbb{P}$ will denote the set of all prime numbers. Further, let $S(n)$ be the set of all prime divisors of a positive integer n. If $w \in \mathbb{Q}$ then by d_w we will denote a denominator of the *lowest terms* fractional representation of w. More specifically, $d_w = n$, if $w = m/n$,

$n \geqslant 1$ and $\gcd(m,n)=1$. We denote $P(w)=S(d_w)$ and $q(w)=\min S(d_w)$ (we assume that $\min \emptyset = 0$). Further, consider, for given $a \in \mathbb{N}$ and $P_0 \subseteq \mathbb{P}$, the set $Q(P_0,a)$ defined by

$$Q(P_0,a) = \{w \in \mathbb{Q};\ P(w) \subseteq P_0 \text{ and } a|d_w\}.$$

We will need the following Lemma.

Lemma 1. *If P_0 is an infinite subset of $\mathbb{P}$, $a \in \mathbb{N}$ and $S(a) \subseteq P_0$, then $Q(P_0,a)$ is a dense subset of $\mathbb{R}$.*

Proof. It suffices to show that the set $X = Q(P_0,a) \cap \mathbb{R}_+$ is dense in $\mathbb{R}_+$. Clearly, $\inf X = 0$ and $\sup X = \infty$. Let $0 < A < B$ and denote $A_0 = \sup\{x \in X;\ x \leqslant A\}$ and $B_0 = \inf\{x \in X;\ x \geqslant B\}$. Then we have $0 < A_0 \leqslant A < B \leqslant B_0 < \infty$ and $(A_0,B_0) \cap X = (A,B) \cap X$. But P_0 is infinite, so there exists $p \in P_0$ such that $\gcd(p,a) = 1$ and

$$\frac{p+a}{p} < \frac{B_0}{A_0}.$$

Clearly, there exists a number $m/n \in X$ satisfying

$$\frac{p}{p+a}A_0 < \frac{m}{n} \leqslant A_0.$$

It follows that

$$\frac{m(p+a)}{np} \in (A_0,B_0) \cap X \subseteq (A,B) \cap X,$$

which ends the proof. □

Now, let $\mathbb{P} = P_1 \cup P_2 \cup P_3 \cup P_4$ be an arbitrary partition of $\mathbb{P}$ into four infinite subsets and let $L = \{p_1 < p_2 < p_3 < \ldots\}$ be an increasing sequence of primes such that $p_{2k} \in P_1$ and $p_{2k-1} \in P_3$, for all $k \in \mathbb{N}$. Next, define a mapping $d : \mathbb{Z} \to L$ by $d(n) = p_{4n+1}$ if $n \geqslant 0$, and $d(n) = p_{-4n-1}$ for $n < 0$. Note, that $d(n) \neq d(m)$ whenever $n \neq m$.

Finally, consider the sets

$$G = \{w \in \mathbb{Q}; P(w) \subseteq P_1 \cup P_3\}$$

and

$$F = \{w \in G; q(w + d(\lfloor w \rfloor)^{-1}) \in P_1 \cup P_2\} \cup \{w \in \mathbb{Q} \setminus G; q(w) \in P_1 \cup P_2\}.$$

Theorem 4. *Define a coloring function $f : \mathbb{Q} \to \{0,1\}$ by $f(x) = 0$, if $x \in F$, and $f(x) = 1$, otherwise. Then for any given rational $\varepsilon > 0$ and any different rational numbers x, y there exists $t \in \mathbb{Q} \cap [0,\varepsilon)$ such that $f(x+t) \neq f(y+t)$. In other words, no two different intervals of $\mathbb{Q}$ are colored similarly with respect to translations.*

Proof. Let $x, y, \varepsilon \in \mathbb{Q}$ be fixed, with $x < y$ and $\varepsilon > 0$. Without loss of generality we can assume that there exists $c \in \mathbb{Z}$ such that $[x, x+\varepsilon) \subseteq [c, c+1)$. Set $k/l = y - x$, where $\gcd(k, l) = 1$.

Case 1: $l = 1$. From the definition of the mapping $d(n)$ it follows that there exists a prime number $s \in P_1$ lying between $d(c)$ and $d(c+k)$. The set $Y = Q(P_1 \cap [s, \infty), s)$ is dense in $\mathbb{R}$ by the Lemma 1, so there exists $w \in Y \cap [x, x+\varepsilon)$. We get easily that $w \in G \cap [c, c+1)$ and $w + k \in G \cap [c+k, c+k+1)$. Now, if $d(c) < d(c+k)$ then $q(w + d(\lfloor w \rfloor)^{-1}) = q(w + d(c)^{-1}) = d(c) \in P_3$ and $q(w + k + d(\lfloor w+k \rfloor)^{-1}) = q(w + k + d(c+k)^{-1}) = s \in P_1$. Hence, $f(w) = 1$ and $f(w+k) = 0$. Otherwise $d(c) > d(c+k)$ and we get $q(w + d(w)^{-1}) = s \in P_1$ and $q(w+k+d(\lfloor w+k \rfloor)^{-1}) = d(c+k) \in P_3$, which gives $f(w) = 0$ and $f(w+k) = 1$. Thus, letting $t = w - x$, we get $f(x+t)) \neq f(y+t)$, and clearly $t \in [0, \varepsilon)$.

Case 2: $q(k/l) \in P_1 \cup P_2$. The set $U = Q(P_4 \cap (l, \infty), 1) \setminus \mathbb{Z}$ is dense in $\mathbb{R}$, by the Lemma 1, so let $w \in U \cap [x, x+\varepsilon)$. Then we have $w \notin G$ and $q(w) \in P_4$. Thus $f(w) = 1$. On the other hand, $f(w + k/l) = 0$, since $w + k/l \notin G$ and $q(w + k/l) = q(k/l) \in P_1 \cup P_2$. Hence $f(x+t) \neq f(y+t)$, with $t = w - x$.

Case 3: $q(k/l) \in P_3 \cup P_4$. Similarly, the set $V = Q(P_2 \cap (l, \infty), 1) \setminus \mathbb{Z}$ is dense in $\mathbb{R}$. So, taking $w \in V \cap [x, x+\varepsilon)$, we have $w \notin G$ and $q(w) \in P_2$. Hence $f(w) = 0$. It is easy to see that $w + k/l \in G$ and $q(w + k/l) = q(k/l) \in P_3 \cup P_4$. Hence $f(w + k/l) = 1$ and once again $f(x+t) \neq f(y+t)$, for $t = w - x$. □

4. Open problems

In our final discussion we would like to propose a more general setting, covering all situations considered so far, and catching many others. Let S be an arbitrary set and denote by $\mathscr{T}$ some fixed family of transformations of S. Suppose further that f is a coloring function of S. Given two subsets $A, B \subseteq S$ are said to be colored *similarly* with respect to $\mathscr{T}$, if there exists a transformation $t \in \mathscr{T}$ such that $t(A) = B$ and $f(a) = f(t(a))$, for every element $a \in A$. In this terminology the theorem of Thue concerns translations on the set of integers, while the Hadwiger–Nelson problem deals with translations of a unit vector in Euclidean spaces.

As one may suspect, translations are not the only one interesting type of transformations one can consider in this vain. It can be proved, for example, that it is possible to color the plane in red and blue, such that no two *topological disks* are colored similarly, with respect to *homeomorphism*s (see [12]). In the discrete case, allowing the full group of permutations of the integers, we get the *strongly non-repetitive* sequences investigated among others by Dekking [8]. Then two blocks are not similar if one of them cannot be obtained from the other by any permutation of its terms. The most difficult case concerned sequences avoiding two adjacent similar blocks. One can see quickly that three colors are not sufficient and Pleasants proved [17] that five will do. The question of Erdős [9], posed in 1961, whether four colors are enough for this property was not answered in the affirmative until 1992 by Keränen [13], who found the appropriate (rather large) substitution.

Our first problem could be regarded as a continuous version of strongly non-repetitive sequences, however, we allow only *measure preserving* transformations instead of arbitrary permutations. Let f be a *measurable* coloring of the real line, i.e., the set of all points in one color is a Lebesgue measurable subset of the line. Denote by $\mu(A)$ the Lebesgue measure of a set A.

Problem 1. *Let f be a measurable coloring of the line by two colors, say red and blue. Denote the sets of red and blue points by R and B, respectively. Is it possible for such a coloring to satisfy*

$$\mu(I_1 \cap B) \neq \mu(I_2 \cap B),$$

for any two different intervals I_1 and I_2 of the same length? In other words, does there exist a measurable 2-coloring of the line, in which no two different line segments are colored similarly, with respect to measure preserving transformations?

Of course, the question can be stated also for any Euclidean space, and, in case of a negative answer, one may increase the number of colors up to ω.

The second problem arose as an attempt of extending Thue's theorem into another direction. Say that a set of integers is *multicolored* if no two of its elements have the same color. In the problem we ask for multicolored arithmetical progressions. In some sense, it can be seen as a dual of the famous Van der Waerden theorem, asserting the existence of arbitrarily long monochromatic arithmetical progressions in any finite coloring of the integers.

Problem 2. *Is it true that there is an absolute constant C such that for every integer $k \geqslant 2$, there is a $(k+C)$-coloring of $\mathbb{N}$, such that for any integer $d \geqslant 1$ each segment of kd consecutive numbers contains a k-term multicolored arithmetic progression of difference d?*

In particular, in such a coloring no two of any k consecutive blocks are colored similarly with respect to translations, so, for $k = 2$ we get the classical non-repetitive sequences of Thue.

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